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NOTE ON THE PROOF OF CONVERGENCE OF THE METHOD OF FINITE ELEMENTS

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A simplification and generalization of the proof presented in [1] is given.

1. Let V be the region occupied by an elastic body subjected to deformation, and V^e be subregions representing finite elements. For simplicity we set, as in [1], $UV^e = V$. The field of displacements $\mathbf{f}^e$ in each subregion is approximated by formula [2]

$$\mathbf{f}^e = \mathbf{N}^e \delta^e \tag{1.1}$$

where $\mathbf{f}^e$ is the displacement vector of points within the element of number e, δ^e is the vector of nodal displacements, and $\mathbf{N}^e$ is a rectangular matrix whose elements are functions of coordinates. The approximation of the displacement throughout region V can be defined by

$$\mathbf{f}_n = \sum_s \sum_{k=1}^{n} \mathbf{f}_{kn}^s \delta_{ks} \tag{1.2}$$

where δ_{ks} are components of the displacement vector of the k-th node and functions $\mathbf{f}_{kn}{}^s$ are piecewise determinate and nonzero only in elements one of whose vertices bears the number k. The system of equations of the method of finite elements is derived by minimizing the functional of energy over the set of functions of the form (1.2). A similar method of solving the problem of minimization of the energy functional was used in [3, 4]. The sequence of approximate solutions converges to the exact (generalized) one, if conditions (1)–(3) of the convergence theorem are satisfied (see Sect. 19 of [3]).

2. Similar results were obtained in [5] for the case when the operator of the boundary value problem contains derivatives of an arbitrary order. Let us briefly consider the contents of [1]. The basic results of the latter can be stated as follows: if function f_{kn}^s contains first degree polynomials and, possibly, terms of higher order, and the maximum diameter of the region in which these functions are defined tends to vanish for $n \to \infty$, then the system of functions $\{f_{kn}^s\}$ is complete in the set of solutions of the boundary value problem of the theory of elasticity, which have continuous derivatives up to and including second order. (Completeness is used here in the meaning of the definition given in [3]).

Investigation of convergence of this method presented in Chapter V of [1] is cumbersome and contains the unproved assumption that the second derivatives of displacements (solutions) are continuous bounded functions in any closed subregion V^e, only if the density of mass forces is by Hölder's definition a continuous function.

3. The statements in Chapter V of [1] can be proved as follows.

The basic result of Chapter IV stated above means that the set of functions $\{f_{kn}^s\}$ is dense in the subset D_A which is the region of determination of operator A of the boundary value problem in the theory of elasticity of the Hilbert space and in H_A which is the energy space of the same problem. Taking into consideration that H_A is the closure of D_A, we conclude that condition (3) of the previously mentioned Mikhlin's theorem is satisfied. We further note that condition (1) of that theorem is satisfied for sets of sequences of coordinate functions $\{f_{kn}^s\}$. This means that operator A belongs to the energy space H_A, when f_{kn}^s are piecewise-polynomial functions which are nonzero in the bounded region of the neighborhood of the node with number k. This follows from the representability of f_{kn}^s in the form of limits with respect to norm H_A of sequences of elements from D_A. The last statement is a corollary of one of the theorems of functional analysis presented in [6]. The fulfillment of condition (2) is checked by the same method as the linear independence (or dependence) of a finite vector set.

The convergence of the method of finite elements is thus established for the case when functions f_{kn}^s satisfy the constraints defined in Sect. 2. Note that the rate of convergence of the selected system of coordinate functions is of the order $\sqrt{h}$, where, as established in [1], h is the greatest of the maximum diameters of the finite elements.

4. The statements of Chapter VI of [1] can be obtained by repeating the reasoning given here and in Chapter IV of [1], taking into consideration that for rods and plates the lattice of elements N_{ij} of matrix N^e the related generalized displacements are: for angles of turn

$$N_{ij}(x_1, x_2, \ldots) = l^e \psi_{ij}(x_1 / l^e, x_2 / l^e, \ldots)$$

for curvatures

$$N_{ij}(x_1, x_2, \ldots) = (l^e)^2 \chi_{ij}(x_1 / l^e, x_2 / l^e, \ldots)$$

where l^e is the maximum diameter of a finite element, and functions ψ_{ij} and χ_{ij}, which are independent of the absolute size of elements, remain bounded for infinite reduction of the size of the latter.

Instead of the tangential field [1] it is necessary to consider in the proof the expansion of displacement functions containing terms which are quadratic with respect to coordinates and, also, to impose on the coordinate functions the condition for these to contain second-order polynomials. Sobolev's imbedding theorem [6] makes it possible to state that in the case of rods and plates the convergence with respect to energy is the mean square convergence of second derivatives, which implies a uniform convergence of the displacements themselves.

5. Since in [5, 7, 8] and others, as well as in [1] the convergence of the method is actually proved in the norm $L_2(V)$ on the assumption that the solution belongs to space $C(V)$, hence all of the above remarks made in relation to [1] also apply to [5, 7, 8]. Stronger and more general results are obtained by using the analogy between the method of finite elements and of the variational-finite difference method [3, 9]. From the point of view of the latter method, the method of finite elements is a procedure of renumbering lattice nodes and unknowns, after which the matrix of the resolving system becomes two-dimensional with a band structure.

The results obtained by Dem'ianovich, Oganesian, Gusman (see [3]), Mikhlin [10] and others with the use of the theory of variational-finite difference method about convergence are directly transferable to the method of finite elements. The restrictions on vertex angles of finite elements, derived in [9, 11, 12] on the basis of the requirement for adequate condition to be specified for the resolving system, are important.

We note in conclusion that conceptually close to the method of finite elements is the method of spline functions [13, 14], whose many estimates and theorems can be transferred to the former. The similarity between the method of finite elements, that of variational-finite difference, and of spline functions was noted and used in [15].

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